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Multicriteria Goal Games[1]

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Abstract. In this paper, we deal with multicriteria matrix games. Different solution concepts have been proposed to cope with these games. Recently, the concept of Pareto-optimal security strategy which assures the property of security in the individual criteria against an opponent's deviation in strategy has been introduced. However, the idea of security behind this concept is based on expected values, so that this security might be violated by mixed strategies when replications are not allowed. To avoid this inconvenience, we propose in this paper a new concept of solution for these games: the *G*-goal security strategy, which includes as part of the solution the probability of obtaining prespecified values in the payoff functions. Thus, attitude toward risk together with payoff values are considered jointly in the solution analysis.

Key Words. Game theory, multicriteria games, solution concepts, goals.

1. Introduction

In this paper, we study two-person games with vector payoff, due to the interesting applications in the analysis of conflict situations when multiple objectives are involved. In fact, the extension of single-criterion games to the multicriteria case provides more realistic models and permits more extensive applications. In practical problems, it is usual that a player deals with not only one criterion, but several criteria which he would like to satisfy. Besides, any competitive situation that can be modeled as a scalar game can be translated into a multicriteria game when more than one objective is present.

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For instance, the duopoly problem with two firms which provide the same product is a two-person scalar game. However, if the two firms are producers of two given homogeneous goods, the model becomes a two-person bicriteria game.

Different solution concepts have been proposed for these games. Shapley (Ref. 1) defined the concept of equilibrium points in games with vector payoff and presented methods for obtaining them. More recently, a similar approach has been adopted (Refs. 2 and 3) using the concept of saddle point for vector-valued function and the notion of efficiency. The interested reader can also see the multicriteria approach in N-person games (Ref. 4) or in multicriteria N-person games under the paradigm of decision dynamics (Ref. 5).

Equilibrium points, as a solution concept for multicriteria games, do not possess the important property of security in the individual criteria against opponent's deviation in strategy, unlike equilibrium saddle points in scalar games. For this reason, Ghose and Prasad (Ref. 6) introduced the concept of Pareto-optimal security strategies (POSS), which is independent of the notion of equilibrium. In Ref. 7, the equivalence between POSS in zero-sum multicriteria matrix games and efficient solutions of a particular vector linear program was established. Multicriteria matrix games have also been studied using as solution concept the utopian-efficient strategy concept (Ref. 8).

We consider a new approach to solve multicriteria games. We assume that, relative to each objective, a goal has been specified by a player, and this player wants to choose a strategy in order to get at least this goal in each objective. Cook (Ref. 9) discusses a similar problem when the player adopts the criterion of minimizing the total expected under achievement of these goals.

We propose goal games, where we define the security level for one of the players, as the probability that prespecified goals fixed by the player might be achieved. Thus, as a part of the solution concept, we study not only the payoff values, but also the probabilities to get them. With this approach, the optimality of a strategy does not depend on the repetition of the game, but it is given by the risk level that the player wants to assume.

A conflict arises immediately between using a pure strategy and using a mixed strategy. The notion of a pure strategy is related to the security to get fixed goals, and the notion of mixed strategy is related to probability distributions. With our approach, a strategy will be chosen taking into account the probability to achieve some goals depending on the player risk position. That is to say, we consider strategies and the probabilities of obtaining them, rather than the expected value given by the classical approach (Refs. 6 and 7). In Ref. 10, using this new solution concept, two-person nonzero sum games were analyzed as bicriteria goal games.

The paper is organized as follows. In Section 2, we formulate and solve the scalar matrix goal game. In Section 3, we generalize this formulation to multicriteria goal games and we define G-goal security strategies. In Section 4, we develop a methodology to obtain these strategies by solving multicriteria linear programs. We prove that the set of these strategies for one of the players coincides with the set of efficient solutions of a multiobjective linear problem. In order to choose a specific strategy in this set, we propose scalar problems associated with the multiobjective problem. Depending on different ways to scalarize the multicriteria game, different efficient solutions will be chosen. In Section 5, we carry out sensitivity analysis in the goals. In addition, a partition on the space of the achievements in payoffs is obtained. Three examples are included clarifying the results in the paper.

2. Matrix Goal Games

Let $A=(a_{ij})$, $1\leq i\leq n$, $1\leq j\leq m$, be the payoff matrix of a two-person zero-sum game. We denote by X and Y the set of mixed strategies for player 1 (PI) and player II (PII) respectively,

$$X=\left\{x\in\mathbb{R}^n, \sum_{i=1}^{n} x_i=1, x_i\geq 0, i=1,\ldots,n\right\}, \tag{1}$$

$$Y=\left\{y\in\mathbb{R}^m, \sum_{j=1}^{m} y_j=1, y_j\geq 0, j=1,\ldots,m\right\}. \tag{2}$$

We analyze the problem under PI point of view.

Let $G\in\mathbb{R}$ be a goal specified by PI. In order to determine the strategies based on the probability to achieve the goal G, we formulate a zero-sum game called matrix G-goal game.

Definition 2.1. The expected payoff of the matrix G-goal game, with goal G and matrix $A=(a_{ij})$, for each strategy pair $x\in X$ and $y\in Y$, is

$$v(x,y)=x^t A_G y, \tag{3}$$

where

$$A_G=(\delta_{ij}), \qquad i=1,\ldots,n, j=1,\ldots,m, \tag{4a}$$

$$\delta_{ij}=\begin{cases}1, & \text{if } a_{ij}\geq G,\\ 0, & \text{otherwise,}\end{cases} \tag{4b}$$

and $v(x,y)$ is the probability to get at least G in the game when PI plays strategy $x\in X$ and PII plays strategy $y\in Y$.

As $v(x, y)$ depends on the strategy that PII plays, we will consider this probability in the worst case; i.e., we assume that PII will choose a strategy $y \in Y$ that gives the minimum value of $v(x, y)$. Then, for each $x \in X$, PI will get

$$v(x) = \min_{y \in Y} v(x, y) = \min_{y \in Y} x^t A_G y = \min_{1 \leq j \leq m} \sum_{i=1}^{n} x_i \delta_{ij}. \tag{5}$$

Definition 2.2. The G-goal security level for PI of a matrix game with matrix $A = (a_{ij})$ is the maximum probability that PI can guarantee to himself for obtaining goal G, irrespective of the actions of PII. It is given by

$$v = \max_{x \in X} v(x) = \max_{x \in X} \min_{y \in Y} v(x, y) = \max_{x \in X} \min_{y \in Y} x^t A_G y. \tag{6}$$

Definition 2.3. A strategy $x \in X$ is a G-goal security strategy (GGSS) for PI if $v = \min_{y \in Y} x^t A_G y$, where v is the G-goal security level of the matrix G-goal game.

The following result characterizes GGSS and gives a procedure to solve matrix goal games.

Theorem 2.1. The G-goal security strategies and the maximum probability to obtain at least goal G are given by the solution of the two-person zero-sum game whose payoff matrix is the matrix A_G.

Proof. For $x = (x_1, x_2, \ldots, x_n) \in X$ and $y = (y_1, y_2, \ldots, y_m) \in Y$, the expected payoff of the zero-sum game with payoff matrix A_G is

$$v(x, y) = x^t A_G y = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \delta_{ij}. \tag{7}$$

For each $i = 1, \ldots, n$, let Y_i be the sum of the y_j's for the columns that have an element equal to 1 in the ith row, i.e.,

$$Y_i = \sum_{j=1}^{m} y_j \delta_{ij}, \qquad i = 1, \ldots, n. \tag{8}$$

The probability of obtaining at least goal G when the players use strategies x and y, respectively, is

$$p = \sum_{i=1}^{n} x_i Y_i = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \delta_{ij} = v(x, y). \tag{9}$$

□

Remark 2.1. If i exists such that $\delta_{ij}=1$, for $j=1, \ldots, m$, then using the ith pure strategy of PI, the probability to get at least goal G is 1. If j exists such that $\delta_{ij}=0$, for $i=1, \ldots, n$, then the probability to get at least goal G is 0, because the jth pure strategy of PII prevents the obtaining of more than this value.

2.1. Decomposition of Goal Space. Previously, we have solved the problem for a known goal G, but we can perform a global analysis when we do not have any information about the goal G that PI wants to achieve. The possible goals that PI can attain are included between the smallest and the biggest element of the payoff matrix A. We call this segment the goal space (GS).

The goal space (GS) can be decomposed into segments such that any goal in a fixed segment can be attained with the same probability. For determining these probabilities in each of these segments, we apply Theorem 2.1 in an orderly way, using sensitivity analysis in linear programming.

Suppose that the matrix A has r different elements. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be those elements ranked in increasing order. The different sets in the partition of the goal space are the segments

$$(\alpha_{i-1}, \alpha_i], \qquad i=2, \ldots, r,$$

and the point α_1.

2.2. Solution Procedure. We consider the matrix A_G for goal $G=\alpha_r$; then, we solve the two-person zero-sum game with payoff matrix A_G. The value of this game is the probability for PI of obtaining at least goal G in the original game for any $G\in(\alpha_{r-1}, \alpha_r]$.

In the following step, we consider the matrix A_G for goal $G=\alpha_{r-1}$. Using the information obtained in the above step (optimal basis), we solve the two-person zero-sum game with this payoff matrix. The value of this game is the probability for PI of obtaining at least goal G in the original game for any $G\in(\alpha_{r-2}, \alpha_{r-1}]$. If we obtain the same solution as that in the above step, then the two segments $(\alpha_{r-1}, \alpha_r]$ and $(\alpha_{r-2}, \alpha_{r-1}]$ may be collapsed into only one, $(\alpha_{r-2}, \alpha_r]$.

This procedure goes on until the first time that we obtain a matrix A_G with all the elements of a row equal to 1. If this happens for goal $G=\alpha_s$, then the probability to attain any goal G such that $G\leq\alpha_s$ is equal to 1.

The following algorithm gives those probability values.

Algorithm 2.1.

Step 1. Make all elements of A_G equal to zero.

Step 2. Determine the position (i, j) corresponding to the biggest element in the matrix A not considered yet. Put 1 in the position (i, j) of matrix A_G.

Step 3. Does A_G have any column with all elements equal to 0? If yes, go to Step 2; if no, go to Step 4.

Step 4. Solve the zero-sum game with payoff matrix A_G. Write down the solution.

Step 5. Does A_G have any row with all elements equal to 1? If yes, go to Step 6; if no, go to Step 2.

Step 6. End.

With this procedure, we obtain the solution set for all possible goals G, and PI will choose among them according to the risk that he wants to take.

Example 2.1. Consider the two-person zero-sum game whose payoff matrix is

$$\begin{bmatrix} 2 & 15 & 8 & 4 \\ 10 & 7 & 2 & 7 \\ 8 & 4 & 10 & 7 \\ 7 & 14 & 4 & 11 \end{bmatrix}.$$

The different sets in the partition of the goal space are

$$2, (2, 4], (4, 7], (7, 8], (8, 10], (10, 11], (11, 14], (14, 15].$$

The probabilities to obtain goals belonging to each set and the corresponding G-goal security strategy set are given in the table below, where ch$\{a, b\}$ is the convex hull of the vectors a, b and X is the mixed strategy space of player I introduced in (1).

Segment	Probability	GGSS set
2	1	X
(2, 4]	1	ch$\{(0, 0, 1, 0), (0, 0, 0, 1)\}$
(4, 7]	2/3	ch$\{(1/3, 1/3, 1/3, 0),$ $(1/3, 0, 1/3, 1/3)\}$
(7, 8]	1/2	$\{(0, 0, 1/2, 1/2)\}$
(8, 10]	1/3	$\{(0, 1/3, 1/3, 1/3)\}$
(10, 11]	0	X
(11, 14]	0	X
(14, 15]	0	X

As we can see, PI will obtain goal $G=7$ with probability equal to 2/3, $\forall x \in \text{ch}\{(1/3, 1/3, 1/3, 0), (1/3, 0, 1/3, 1/3)\}$, $\forall y \in Y$. However, the probability to get goal $G=9$ is equal to 1/3 for $x=(0, 1/3, 1/3, 1/3)$, $\forall y \in Y$.

3. Multicriteria Goal Games: Model and Definitions

In this section, we extend the results obtained for the scalar game to the multicriteria matrix game. See Refs. 6 and 7 for further details.

Let $A=(a_{ij})$ be the payoff matrix of a multicriteria zero-sum game, with $a_{ij}=(a_{ij}(1), a_{ij}(2), \ldots, a_{ij}(k)) \in \mathbb{R}^k$, $1 \leq i \leq n$, $1 \leq j \leq m$, which leads to $n \times m$ matrices

$$A(s)=(a_{ij}(s)), \qquad 1 \leq s \leq k, \qquad 1 \leq i \leq n, \qquad 1 \leq j \leq m. \tag{10}$$

The mixed strategies spaces for PI and PII are, respectively,

$$X=\left\{x \in \mathbb{R}^n, \sum_{i=1}^{n} x_i=1, x_i \geq 0, i=1, \ldots, n\right\},$$

$$Y=\left\{y \in \mathbb{R}^m, \sum_{j=1}^{m} y_j=1, y_j \geq 0, j=1, \ldots, m\right\}.$$

Let $G=(G_1, \ldots, G_k)$ be a vector of goals specified by PI. Each component of G is a goal for the corresponding scalar game.

Definition 3.1. The expected payoff of the goal game with goal $G=(G_1, \ldots, G_k)$ and matrices $A(s)=(a_{ij}(s))$, $s=1, \ldots, k$, for each strategy pair $x \in X$ and $y \in Y$, is

$$v^G(x, y)=x^t A_G y=(v_1^G(x, y), \ldots, v_k^G(x, y)), \tag{11}$$

where

$$v_s^G(x, y)=x^t A_G(s) y, \qquad s=1, \ldots, k, \tag{12}$$

$$A_G(s)=(\delta_{ij}^s), \qquad 1 \leq s \leq k, \qquad 1 \leq i \leq n, \qquad 1 \leq j \leq m, \tag{13}$$

$$\delta_{ij}^s=\begin{cases} 1, & \text{if } a_{ij}(s) \geq G_s, \quad \forall s=1, \ldots, k, \\ 0, & \text{if } a_{ij}(s) < G_s, \quad \forall s=1, \ldots, k. \end{cases} \tag{14}$$

Every strategy $x \in X$ defines goal security levels for each scalar game induced by the vector payoff game.

Definition 3.2. The G-goal security level vector for PI of a multicriteria matrix G-goal game with goal $G=(G_1, \ldots, G_k)$ and matrices $A(s)=(a_{ij}(s))$,

$s=1,\ldots,k$, for each $x\in X$, is

$$v^G(x)=(v_1^G(x),\ldots,v_k^G(x)),\tag{15}$$

where

$$\begin{aligned} v_s^G(x) &= \min_{y\in Y} v_s^G(x,y) = \min_{y\in Y} x^t A_G(s) y \\ &= \min_{1\le j\le m} \sum_{i=1}^n x_i \delta_{ij}^s, \qquad s=1,\ldots,k; \end{aligned}\tag{16}$$

here, $v_s^G(x)$, $s=1,\ldots,k$, is the probability to achieve at least G_s in each scalar game when PI chooses strategy x.

Notice that, for a given strategy $x\in X$, the security level $v_s^G(x)$, $s=1,\ldots,k$, might be obtained from different strategies $y\in Y$.

Now, we establish a new solution concept for vector payoff games, based on goal security levels.

Definition 3.3. A strategy $x^*\in X$ is a G-goal security strategy (GGSS) for PI if there is no $x\in X$ such that $v^G(x^*)\le v^G(x)$, $v^G(x^*)\ne v^G(x)$.

As we have a multicriteria game, the solution concept is based on Pareto optimality; i.e., any improvement of a component of $v^G(x^*)$ can be achieved only if another component gets worse values.

In order to obtain the whole set of GGSS, we propose a characterization using multicriteria linear programming. To this end, in Section 4, we identify GGSS with Pareto-efficient solutions of a particular multiple-objective linear problem.

4. Determination of G-Goal Security Strategies

We consider the following multiple-objective linear problem, which we call the G-goal game linear multicriteria problem:

$$(\text{GLMP})_G \quad \max \quad v_1,\ldots,v_k,\tag{17a}$$

$$\text{s.t.} \quad x^t A_G(s)\ge(v_s,\ldots,v_s), \qquad s=1,\ldots,k,\tag{17b}$$

$$\sum_{i=1}^n x_i=1,\tag{17c}$$

$$x\ge 0.\tag{17d}$$

Theorem 4.1. A strategy $x^* \in X$ is a GGSS and $v^* = (v_1^*, \ldots, v_k^*)$ is its G-goal security level vector iff (v^*, x^*) is an efficient solution of problem (GLMP).

Proof. Let x^* be a GGSS. Then, there is no $x \in X$ such that

$$v^G(x^*) \leq v^G(x), \qquad v^G(x^*) \neq v^G(x).$$

From (14), this is equivalent to

$$(\min x^T A_G(1), \ldots, \min x^T A_G(k)) \geq (\min x^{*\prime} A_G(1), \ldots, \min x^{*\prime} A_G(k)),$$

$$(\min x' A_G(1), \ldots, \min x' A_G(k)) \neq (\min x^{*\prime} A_G(1), \ldots, \min x^{*\prime} A_G(k)).$$

Hence, x is an efficient solution of the problem

$$\max_{x \in X} (\min x' A_G(1), \ldots, \min x' A_G(k)),$$

and this problem is equivalent to

$$\max \quad v_1, \ldots, v_k, \tag{18a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{18b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{18c}$$

$$x \geq 0. \tag{18d}$$

Conversely, suppose that an efficient solution (v^*, x^*) of $(\text{GLMP})_G$ is not a GGSS. Then, there exists $\bar{x} \in X$ such that

$$(\min \bar{x}' A_G(1), \ldots, \min \bar{x}' A_G(k))$$
$$\geq (\min x^{*\prime} A_G(1), \ldots, \min x^{*\prime} A_G(k)),$$
$$(\min \bar{x}' A_G(1), \ldots, \min \bar{x}' A_G(k))$$
$$\neq (\min x^{*\prime} A_G(1), \ldots, \min x^{*\prime} A_G(k)).$$

Taking $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_k)$, where

$$\bar{v}_s = \min \bar{x}' A_G(s), \qquad s = 1, \ldots, k,$$

the vector $(\bar{v}, \bar{x})$ is a feasible solution of $(\text{GLMP})_G$ dominating (v^*, x^*); this is a contradiction, because (v^*, x^*) is an efficient solution of $(\text{GLMP})_G$. □

The characterization given by Theorem 4.1 allows one to obtain all GGSS solving the multiobjective problem $(\text{GLMP})_G$. Besides, we can consider different scalarization methods for choosing one of them (Ref. 7).

(A) Firstly, we consider the scalarization given through weighting the scalar linear problem $P(\lambda)$ associated with $(GLMP)_G$,

$$(P(\lambda))\quad \max \quad \lambda_1 v_1 + \cdots + \lambda_k v_k, \tag{19a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{19b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{19c}$$

$$x \geq 0, \tag{19d}$$

$$\text{where} \quad \lambda \in \Lambda^0 = \left\{ \lambda \in \mathbb{R}^k \,\middle|\, \lambda_s > 0, \sum_{s=1}^{k} \lambda_s = 1 \right\}.$$

The next result gives a characterization of GGSS as solutions of problem $P(\lambda)$.

Theorem 4.2. A strategy $x^* \in X$ is a GGSS and $v^* = (v_1^*, \ldots, v_k^*)$ is its G-goal security level vector iff exists $\lambda^* \in \Lambda^0$ such that (v^*, x^*) is an optimal solution of problem $P(\lambda)$.

Proof. This follows from the characterization of GGSS given in Theorem 4.1 and the equivalence between efficient solutions of a multiobjective linear problem and the solutions of the associated weighted-sum problems. □

Each component λ_s of the parameter $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda^0$ can be interpreted as the relative importance that PI assigns to the corresponding scalar game with matrix $A(s)$. Thus, if PI sets up fixed values for λ_s, the objective function of problem $P(\lambda)$ is perfectly determined. If PI chooses $\lambda_s = G_s$, $s = 1, \ldots$, k, this function is the expected value of the goals G_s.

In this case, PI might choose a G-goal security strategy x^* that gives the biggest expected value, i.e., the optimal solution of the following scalar linear problem:

$$\max \quad G_1 v_1 + \cdots + G_k v_k, \tag{20a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{20b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{20c}$$

$$x \geq 0. \tag{20d}$$

(B) We now consider the scalarization given through the weighted maximin problem associated with $(\mathrm{GLMP})_G$:

$$(\mathrm{WMP}(\omega)) \quad \max\min \quad \omega_1 v_1, \ldots, \omega_k v_k, \tag{21a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{21b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{21c}$$

$$x \geq 0, \tag{21d}$$

$$\text{with} \quad \omega \in W = \{\omega \in \mathbb{R}^k / \omega > 0\}. \tag{21e}$$

This problem can be written equivalently

$$\max \quad z, \tag{22a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{22b}$$

$$\omega_s v_s \geq z, \qquad s = 1, \ldots, k, \tag{22c}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{22d}$$

$$x \geq 0, \tag{22e}$$

$$\text{with} \quad z \in \mathbb{R}. \tag{22f}$$

The following theorem states that the solution of $\mathrm{WMP}(\omega)$ with $\omega > 0$ is both a necessary and sufficient condition for a strategy to be a GGSS for PI.

Theorem 4.3. A strategy $x \in X$ is a GGSS for PI if and only if (v^*, x^*) is an optimal solution to $\mathrm{WMP}(\omega)$ with $\omega \in W$.

Proof. In Ref. 11, it is established that (v^*, x^*) is an efficient solution to (GLMP) iff there is a $\omega^0 \in W$ such that (v^*, x^*) is an optimal solution to $\mathrm{WMP}(\omega^0)$; from Theorem 4.1, that is equivalent to a GGSS for PI. □

In this case, if PI chooses $\omega_s = G_s$, $s = 1, \ldots, k$, the optimal solution of problem

$$\max\min \quad G_1 v_1, \ldots, G_k v_k, \tag{23a}$$

$$\text{s.t.} \quad x' A_G(s) \geq (v_s, \ldots, v_s), \qquad s = 1, \ldots, k, \tag{23b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{23c}$$

$$x \geq 0, \tag{23d}$$

determines a G-goal security strategy that shares out the risk to obtain goals G_s, $s = 1, \ldots, k$, among all them.

Example 4.1. Consider the following payoff matrix proposed in Ref. 12:

$$\begin{bmatrix} (1,3) & (2,1) \\ (3,1) & (1,2) \\ (1,1) & (3,3) \end{bmatrix}.$$

Let $G = (3, 2)$ be a vector of goals fixed by PI. The matrices $A_G(1)$ and $A_G(2)$ induced by goal G are

$$A_G(1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_G(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

In order to get all the G-goal security strategies for PI, we solve the following linear multiobjective problem:

$$\max \quad v_1, v_2, \tag{24a}$$

$$\text{s.t.} \quad x_2 \geq v_1, \tag{24b}$$

$$x_3 \geq v_1, \tag{24c}$$

$$x_1 \geq v_2, \tag{24d}$$

$$x_2 + x_3 \geq v_2, \tag{24e}$$

$$\sum_{i=1}^{3} x_i = 1, \tag{24f}$$

$$x \geq 0, \qquad v_1, v_2 \in \mathbb{R}. \tag{24g}$$

The extreme efficient solutions are

$$(v^1, x^1) = (1/4, 1/2; 1/2, 1/4, 1/4),$$

$$(v^2, x^2) = (1/2, 0; 0, 1/2, 1/2),$$

and the G-goal security strategy set for PI is

$$\text{ch}\{(1/2, 1/4, 1/4), (0, 1/2, 1/2)\}.$$

The G-goal security strategy that gives the biggest expected value of goal G is given by

$$v^1 = (1/4, 1/2),$$
$$x^1 = (1/2, 1/4, 1/4),$$

the optimal solution of the following linear problem:

$$\max \quad 3v_1 + 2v_2, \tag{25a}$$
$$\text{s.t.} \quad x_2 \geq v_1, \tag{25b}$$
$$x_3 \geq v_1, \tag{25c}$$
$$x_1 \geq v_2, \tag{25d}$$
$$x_2 + x_3 \geq v_2, \tag{25e}$$
$$\sum_{i=1}^{3} x_i = 1, \tag{25f}$$
$$x \geq 0, \qquad v_1, v_2 \in \mathbb{R}. \tag{25g}$$

The GGSS that shares out the risk to obtain goals $G = (3, 2)$ is

$$x = (11/25, 7/25, 7/25), \; v = (7/25, 11/25),$$

which is the optimal solution of the following linear problem:

$$\max \quad z, \tag{26a}$$
$$\text{s.t.} \quad x_2 \geq v_1, \tag{26b}$$
$$x_3 \geq v_1, \tag{26c}$$
$$x_1 \geq v_2, \tag{26d}$$
$$x_2 + x_3 \geq v_2, \tag{26e}$$
$$3v_1 \geq z, \tag{26f}$$
$$2v_2 \geq z, \tag{26g}$$
$$\sum_{i=1}^{3} x_i = 1, \tag{26h}$$
$$x \geq 0, \qquad v_1, v_2, z \in \mathbb{R}. \tag{26i}$$

5. Sensitivity Analysis in the Goals

In Section 4, we have obtained a G-goal security strategy and its G-goal security level vector solving a multiobjective linear problem. Now, we want to determine whether an efficient solution (v^*, x^*) for this problem remains efficient after changing the goals G_s. When goal G is not known, we will obtain the efficient solution set for any value of G. The set of all possible goals that PI can attain in the game, called the goal space (GS), can be decomposed into regions such that any goal within one of these regions can be obtained with the same probability.

We consider two cases. In the first, we assume that the goals $G=(G_1, \ldots, G_k)$ increase to $G'=(G'_1, \ldots, G'_k)$; in the second we assume that the goals $G=(G_1, \ldots, G_k)$ decrease to $G'=(G'_1, \ldots, G'_k)$.

(C1) If we increase each goal G_s to G'_s, $s=1, \ldots, k$, the corresponding matrix $A_{G'}(s)$ induced by G'_s, $s=1, \ldots, k$, has more zero elements than the matrix $A_G(s)$, $s=1, \ldots, k$. For this reason, the feasible set of the new linear problem $(\text{GLMP})_{G'}$ is smaller. Hence, if (v^*, x^*) remains feasible for the problem associated to goals G', it will be efficient for that problem.

We consider the matrices

$$M(s)=(m_{ij}(s)), \qquad 1 \le s \le k,\ 1 \le i \le n,\ 1 \le j \le m,$$

whose elements are

$$m_{ij}(s)=\begin{cases} 1, & \text{if } G_s \le a_{ij}(s) < G'_s, \quad \forall s=1, \ldots, k, \\ 0, & \text{otherwise}, \quad \forall s=1, \ldots, k. \end{cases} \tag{27}$$

Theorem 5.1. Let (v^*, x^*) be an efficient solution of problem $(\text{GLMP})_\gamma$. If

$$\sum_{i=1}^{n} x_i^* m_{ij}(s) \le h_j^*(s), \qquad j=1, \ldots, m,\ \forall s=1, \ldots, k,$$

where $h_j^*(s)$, $j=1, \ldots, m$, are the slack variables of the efficient solution in problem $(\text{GLMP})_G$, then (v^*, x^*) is an efficient solution of problem $(\text{GLMP})_{G'}$.

Proof. We can express

$$A_{G'}(s)=A_G(s)-M(s), \qquad s=1, \ldots, k.$$

If (v^*, x^*) is an efficient solution of problem $(\text{GLMP})_G$, then

$$x^{*\prime} A_G(s) \ge (v_s^*, \ldots, v_s^*), \qquad s=1, \ldots, k, \tag{28a}$$

$$\sum_{i=1}^{n} x_i^* = 1, \tag{28b}$$

$$x^* \ge 0. \tag{28c}$$

Because we assume by hypothesis that

$$\sum_{i=1}^{n} x_i^* m_{ij}(s) \leq h_j^*(s), \qquad j=1,\ldots,k, \forall s=1,\ldots,k,$$

where

$$h_j^*(s) = \sum_{i=1}^{n} x_i^* \delta_{ij}^s - v_s^*, \qquad j=1,\ldots,m,$$

these expressions can be rewritten as

$$x^{*\prime} M(s) \leq x^{*\prime} A_G(s) - (v_s^*,\ldots,v_s^*), \qquad s=1,\ldots,k,$$

which means that

$$\begin{aligned}(v_s^*,\ldots,v_s^*) &\leq x^{*\prime} A_G(s) - x^{*\prime} M(s)\\ &= x^{*\prime}(A_G(s) - M(s))\\ &= x^{*\prime} A_{G'}(s), \qquad \forall s=1,\ldots,k,\end{aligned}$$

and

$$\sum_{i=1}^{n} x_i^* = 1, \qquad x^* \geq 0,$$

which implies that (v^*, x^*) is an efficient solution of problem $(\text{GLMP})_{G'}$.
□

(C2) We now suppose that the goals G_s, $s=1,\ldots,k$, decrease to the new values G_s', $s=1,\ldots,k$. In this case, the corresponding matrix $A_{G'}(s)$ induced by G_s', $s=1,\ldots,k$, has more elements equal to 1 than the matrix $A_G(s)$, $s=1,\ldots,k$. Then, the feasible set of the new problem increases. For this reason, if (v^*, x^*) is an efficient solution for the problem with goals G_s, it will remain a feasible solution for the problem with goals G_s', but may not be an efficient solution. To check if (v^*, x^*) is an efficient solution for the new problem, subproblem testing can be used.

Let $A_{G'}(s)$ be the matrix induced by G_s', $s=1,\ldots,k$. The new problem is

$$(\text{GLMP})_{G'} \quad \max \quad v_1,\ldots,v_k, \tag{29a}$$

$$\text{s.t.} \quad x' A_{G'}(s) \geq (v_s,\ldots,v_s), \qquad s=1,\ldots,k, \tag{29b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{29c}$$

$$x \geq 0. \tag{29d}$$

We can express

$$A_{G'}(s) = A_G(s) + M(s), \qquad s = 1, \dots, k,$$

where $M(s) = (m_{ij}(s))$ is a matrix whose elements are

$$m_{ij}(s) = \begin{cases} 1, & \text{if } G'_s \leq a_{ij}(s) < G_s, \quad \forall s = 1, \dots, k, \\ 0, & \text{otherwise}, \qquad\qquad\quad \forall s = 1, \dots, k. \end{cases} \tag{30}$$

Problem (29) can be written as

$$(\text{GLMP})_{G'} \quad \max \quad v_1, \dots, v_k, \tag{31a}$$

$$\text{s.t.} \quad x'(A_G(s) + M(s)) \geq (v_s, \dots, v_s), \qquad s = 1, \dots, k, \tag{31b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{31c}$$

$$x \geq 0. \tag{31d}$$

Theorem 5.2. Let (v^*, x^*) be an efficient solution of problem $(\text{GLMP})_G$. If the scalar linear problem

$$\max \quad t_1 + t_2 + \cdots t_k, \tag{32a}$$

$$\text{s.t.} \quad x'(A_G(s) + M(s)) \geq (v_s, \dots, v_s), \qquad s = 1, \dots, k, \tag{32b}$$

$$v_s - t_s = v_s^*, \qquad s = 1, \dots, k, \tag{32c}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{32d}$$

$$x \geq 0, \qquad t_s \geq 0, \qquad s = 1, \dots, k, \tag{32e}$$

has an optimal value equal to zero, then (v^*, x^*) is an efficient solution for $(\text{GLMP})_{G'}$.

Proof. If the optimal objective function value is zero, then $t_i = 0$, $\forall i = 1, \dots, k$. Using subproblem testing for efficient points (see Ref. 13), this means that solution (v^*, x^*) cannot be improved. □

On the other hand, we can obtain the efficient solution set of problem $(\text{GLMP})_G$ for any possible goal G in the goal space. For determining these sets, we apply Theorem 4.1 in an orderly way for all goals. Using the information obtained in each step (efficient basis), we can develop an iterative method in order to get the efficient solutions of the new problem.

Example 5.1. Consider the payoff matrix of Example 4.1. The goal space for this game is

$$\mathrm{GS}=\{G=(G_1, G_2)| 1\le G_1\le 3,\ 1\le G_2\le 3\}.$$

Similarly to the scalar game, we consider the different elements of the matrices $A(1)$ and $A(2)$ ranked in increasing order. Therefore, the different regions in the partition of the goal space are

$$R_{11}=\{(1, 1)\},$$

$$R_{1j}=\{1\}\times(j-1, j], \qquad j=2, 3,$$

$$R_{i1}=(i-1, i]\times\{1\}, \qquad i=2, 3,$$

$$R_{ij}=(i-1, i]\times(j-1, j], \qquad i, j=2, 3.$$

Notice that these regions correspond to rectangles, segments, and one point. We denote by $S_{(i,j)}$ the efficient solution set of problem $(\mathrm{GLMP})_G$ for any goal $G\in R_{ij}$, $i, j=1, 2, 3$. In our example, these sets are

$$S_{(3,3)}=\mathrm{ch}\{(1/3, 1/3; 1/3, 1/3, 1/3), (0, 1/2; 1/2, 0, 1/2)\}$$
$$\cup\,\mathrm{ch}\{((1/3, 1/3; 1/3, 1/3, 1/3), (1/2, 0; 0, 1/2, 1/2)\}$$
$$\cup\,\mathrm{ch}\{(0, 1/2; 1/2, 0, 1/2), (1/2, 0; 0, 1/2, 1/2)\},$$

$$S_{(3,2)}=\mathrm{ch}\{(1/4, 1/2; 1/2, 1/4, 1/4), (1/2, 0; 0, 1/2, 1/2)\},$$

$$S_{(3,1)}=\{(1/2, 1; 0, 1/2, 1/2)\},$$

$$S_{(2,3)}=\mathrm{ch}\{(1/2, 1/4; 1/4, 1/2, 1/4), (0, 1/2; 1/2, 0, 1/2)\},$$

$$S_{(2,2)}=\{(1/2, 1/2; 1/2, 1/2, 0)\},$$

$$S_{(2,1)}=\mathrm{ch}\{(1/2, 1; 1/2, 1/2, 0), (1/2, 1; 0, 1/2, 1/2)\},$$

$$S_{(1,3)}=\{(1, 1/2; 1/2, 0, 1/2)\},$$

$$S_{(1,2)}=\mathrm{ch}\{(1, 1/2; 1/2, 0, 1/2), (1, 1/2; 1/2, 1/2, 0)\},$$

$$S_{(1,1)}=\mathrm{ch}\{(1, 1; 1, 0, 0), (1, 1; 0, 1, 0), (1, 1; 0, 0, 1)\}.$$

6. Conclusions

A new solution concept has been introduced for both scalar and multicriteria matrix games. This concept is based on two basic rationality principles: (i) security in the individual criteria against opponent's deviation in strategy; and (ii) measurability of the risk attitude in mixed strategies when replications are not allowed. Using prespecified goals for criteria, we consider as solution not only the strategy played by the player, but also the probability of obtaining at least such goal values. Therefore, this concept permits the players to measure their attitude towards risk by means of the probability that they have for obtaining the different outcomes of the games.

This approach can be viewed as a refinement of the concept of Pareto-optimal security strategy; see Refs. 6, 7, 14. This extension consists of adding information about the probabilities of different outcomes to the idea of security behind any Pareto-optimal security strategy.

A methodology to obtain the whole set of GGSS is developed. We have shown that all these strategies, together with their associated probabilities, can be obtained as Pareto-efficient solutions of a particular multiobjective linear problem.

Finally, we want to point out that these concepts can be applied beyond this framework. In fact, they have been used to develop a new solution concept for two-person nonzero-sum games. It consists of considering any bimatrix game as a bicriteria matrix game and then applying the approach presented in this paper. More details on this subject can be found in Ref. 10.

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